Classification of vector bundles on the curve

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3ème cours

$E|\mathbb{Q}_p$. $F|\mathbb{F}_q$ alg. closed

$X_E / \mathrm{Spec}\, E$ the curve

Construction of some vector bundles

$$X_E = \mathrm{Proj}\Big(\underbrace{P_{E,\pi}}_{\bigoplus_{d \geq 0} B_E^{\varphi_E = \pi^d}} \Big)$$

$$\rightsquigarrow \mathcal{O}_{X_E}(d) = \underbrace{\widetilde{P_{E,\pi}[d]}}_{\text{graded } P\text{-module}} \quad \text{line bundle}/X_E$$

Rem: X_E does not depend canonically on the choice of π but $\mathcal{O}_{X_E}(1)$ does: another choice of uniformizing element leads to an isomorphic line bundle but the iso. is not canonical.

$*X_E$ "Complete" $(\deg \operatorname{div} f = 0)$

$$\Rightarrow \deg : \operatorname{Pic}(X_E) = \operatorname{Div}(X_E)/\sim \longrightarrow \mathbb{Z}$$

$$\uparrow$$

$$\operatorname{div}(E(X_E)^*)$$

$\rightsquigarrow \deg(\mathcal{E}) := \deg(\det \mathcal{E})$ for $\mathcal{E}$ a v.b.

$\mu = \frac{\deg}{\operatorname{rk}}$ $\rightsquigarrow$ Harder-Narasimhan reduction theory

Prop. $\operatorname{Pic}(X_E) \xrightarrow[\sim]{\deg} \mathbb{Z}$ i.e. $\operatorname{Pic}(X_E) = \langle \mathcal{O}_{X_E}(1) \rangle$

$\rightarrow$ Consequence of $X_E \setminus \{\infty\} = \operatorname{Spec}(\text{P.I.D.})$. $\square$

$E_h|E$ unramified degree h extension of E.

$\mathrm{Gal}(E_h|E) = \mathbb{Z}/h\mathbb{Z}$

$$\begin{tikzcd} X_{E_h} = X_E \otimes_E E_h \arrow[d, "\pi_h"] \\ X_E \end{tikzcd} \Big) \ \mathbb{Z}/h\mathbb{Z} \text{ - Galois cover}$$

Via GAGA: $\varphi_{E_h} = \varphi_E^h$, $W_{O_{E_h}} = W_{O_E}$, replacing E by E_h does not change $Y_{E_h} = Y_E$, it changes the Frobenius

$$\begin{tikzcd} X_{E_h}^{ad} = Y_E / \varphi_E^{h\mathbb{Z}} \arrow[d, "\pi_h^{ad}"] \\ X_E^{ad} = Y_E / \varphi_E^{\mathbb{Z}} \end{tikzcd}$$

$$\begin{tikzcd} (X_{E_h})_{h\geq 1} \arrow[d] \\ X_E \end{tikzcd} \Big) \ \hat{\mathbb{Z}} \text{ Galois pro-cover}$$

analogous to

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{C}} \arrow[d] \\ \mathbb{P}^1_{\mathbb{R}} \end{tikzcd} \Big) \ \mathbb{Z}/2\mathbb{Z}$$

$$\pi_{Eh}^{*}\,\mathcal{O}_{X_E}(d) = \mathcal{O}_{X_{Eh}}(hd)$$

Def: $\lambda \in \mathbb{Q}$, $\lambda = \frac{d}{h}$, $h \geq 1$, $d \in \mathbb{Z}$ and $(d,h) = 1$.

$$\mathcal{O}_{X_E}(\lambda) = \underbrace{\pi_{h*}\,\mathcal{O}_{X_{E_h}}(d)}_{\text{rank } h \text{ degree } d}$$

Semi-stable slope λ

↳ pushforward of a semi-stable v.b. by a finite étale Galois cover = semi-stable.

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) = \bigoplus_{\text{finite}} \mathcal{O}(\lambda+\mu)$$

$$\mathcal{O}(\lambda)^{\vee} = \mathcal{O}(-\lambda)$$

$$\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} H^0(X, \mathcal{O}(\mu-\lambda))$$

$$= 0 \text{ if } \lambda > \mu$$

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Since if $r = \frac{d}{h}$ $H^0(X_E, \mathcal{O}(r))$

$$\parallel$$

$$H^0(X_{E_h}, \mathcal{O}_{X_{E_h}}(d)) = 0 \text{ if } d < 0.$$

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} H^1(X, \mathcal{O}(\mu - \lambda))$$

$$= 0 \text{ if } \lambda \leq \mu.$$

Since $r = \frac{d}{h}$ $H^1(X_E, \underbrace{\mathcal{O}_{X_E}(r)}_{\pi_{E_h *}\mathcal{O}_{X_{E_h}}(d)}) = H^1(X_{E_h}, \mathcal{O}_{X_{E_h}}(d))$

$$= 0 \text{ if } d \geq 0$$

Th: (1) $\lambda \in \mathbb{Q}$. Then any slope λ semi-stable vector bundle is isomorphic to a direct sum of $\mathcal{O}_X(\lambda)$

(2) The H.N. filtration of a vector bundle is split.

(3)

$$\{\lambda_1 \geq \dots \geq \lambda_n \,/\, \lambda_i \in \mathbb{Q}, n \in \mathbb{N}\} \xrightarrow{\sim} \mathrm{Bun}_X/\sim$$

$$(\lambda_i)_i \longmapsto \left[\bigoplus_i \mathcal{O}_X(\lambda_i)\right]$$

Rem: (1) + (2) $\Leftarrow\Rightarrow$ (3)

Moreover (1) $\Rightarrow$ (2) via the computation of $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu))$ $= 0$ if $\lambda \leq \mu$

* In particular

$$\mathrm{Vect}_E \xrightarrow{\sim} \mathrm{Bun}_X^{ss,0} = \text{abelian category of slope } 0 \text{ semi-stable v.b.}/X$$

$$V \longmapsto V \otimes_E \mathcal{O}_X$$

$$H^0(X, \mathcal{E}) \longleftarrow \mathcal{E}$$

- A v.b./X is trivial iff it is semi-stable of slope 0.

* More generally, $\mathrm{End}(\mathcal{O}(\lambda)) = D_\lambda^{op}$

D_λ = division algebra$/E$ wrt. invariant λ.

$$\begin{cases} \mathrm{Vect}_{D_\lambda} \xrightarrow{\sim} \mathrm{Bun}_X^{ss,\lambda} \\ V \longmapsto \mathcal{O}(\lambda) \otimes_{D_\lambda} V \end{cases}$$

From Isocrystals to vector bundles

$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma$

$\varphi\text{-Mod}_{\breve{E}}$ = abelian category of isocrystals

= semi-simple (Dieudonné–Manin)

$$\varphi\text{-Mod}_{\breve{E}} = \bigoplus_{\lambda \in \mathbb{Q}}^{\perp} \underbrace{\varphi\text{-Mod}_{\breve{E}}^{\lambda}}_{\text{Isoclinic with slope } \lambda}$$

$\forall \lambda$ $\exists!$ simple object $N_\lambda = \langle e, \varphi(e), \dots, \varphi^{h-1}(e) \rangle$, $\lambda = \frac{d}{h}$

with $\varphi^h(e) = \pi^d e$.

$$\text{Matrix of } \varphi = \left(\begin{array}{ccc|c} 0 & & & \pi \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{array}\right)$$

Functor $\otimes$-exact:

$$\begin{array}{ccc} \varphi\text{-}\mathrm{Mod}_{\breve{E}} & \longrightarrow & \mathrm{Bun}_X \\ (D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi) \\ & & \| \\ & & \underbrace{\Big[\bigoplus_{d\geq 0} \big(D \otimes_{\breve{E}} B\big)^{\varphi\otimes\varphi = \pi^d} \Big]}_{\text{graded } P\text{-module}}^{\sim} \end{array}$$

Via GAGA: $\mathcal{E}(D,\varphi)^{ad}$ = v.b. on $Y/\varphi^{\mathbb{Z}}$ corresponding to the φ-equivariant v.b.

$$\big(\underbrace{D \otimes_{\breve{E}} \mathcal{O}_Y}_{\text{trivial v.b. } /Y \text{ with fiber } D}, \varphi\otimes\varphi\big)$$

If $(D,\varphi) = \langle e, \varphi(e), \dots, \varphi^{h-1}(e) \rangle$ with $\varphi^h(e) = \pi^d e$

$\lambda = \frac{d}{h}$
$(d,h) = 1$

$\mathcal{E}(D,\varphi) = \mathcal{O}_X(-\lambda)$.

Thus, via Dieudonné-Manin the classification theorem says that $(D,\varphi) \mapsto \mathcal{E}(D,\varphi)$ is essentially surjective.

Periods of p-divisible groups

Main tool in the classification theorem. $E = \mathbb{Q}_p$ to simplify.

- $C|\mathbb{Q}_p$ algebraically closed with $C^\flat = F$.
- $\infty \in |X|$ with $k(\infty) = C$.

Purpose: $BT_{\mathcal{O}_C}$ = Barsotti-Tate p-div. groups $/\mathcal{O}_C$

Explain: $\exists$ functor

$$\begin{aligned} BT_{G_0} &\longrightarrow \text{Modifications of vector bundles} \\ H &\longmapsto \Big[0 \to \underbrace{V_p(H)\otimes \mathcal{O}_X}_{\substack{\text{trivial v.b.} \\ \text{with fiber } V_p(H)}} \to \mathcal{E}_H \to i_{\infty *}\,\mathrm{Lie}\,H[\tfrac{1}{p}] \to 0\Big] \end{aligned}$$

with $\mathcal{E}_H \simeq \mathcal{E}(\underbrace{D, p^{-1}\varphi}_{\substack{\text{Covariant isocrystal of the} \\ \text{reduction of } H.}})$

Periods in characteristic p

Dieudonné-Manin classification:

$k | \mathbb{F}_p$ perfect field.

$BT_k =$ p-div. groups$/k$

$$\left\{\begin{array}{l} BT_k \xrightarrow{\sim} \{\text{Dieudonné-crystals}\} = \{(M, F, V)\} \\ H \longmapsto \underbrace{\mathbb{D}(H)} \end{array}\right.$$

$\mathbb{D}(H)$: contravariant Dieudonné crystal.

M free $W(k)$-mod. finite rk.
F σ-linear endo
V σ^{-1}-linear endo
$FV = VF = p$.

The Covectors:

$W_n = \{[x_0, \dots, x_{n-1}]\} = W / V^n W$ truncated Witt vectors of lenght n

$\shortparallel$

affine unipotent group scheme isomorphic to $\mathbb{A}^n_k$.

$$W_n \overset{V}{\hookrightarrow} W_{n+1} \overset{V}{\hookrightarrow} W_{n+2} \hookrightarrow \cdots$$

$$[x_0, \dots, x_{n-1}] \longmapsto [0, x_0, \dots, x_{n-1}]$$

$$CW^u := \varinjlim_{m\geq 1} W_m$$ Unipotent Witt Covectors

$$= \{ [x_n]_{n\leq 0} \mid x_n = 0 \text{ for } n \ll 0 \}$$

Here $[x_n]_{n\leq 0} + [y_n]_{n\leq 0} = [z_n]_{n\leq 0}$

with $z_n = P_b(x_{n-b}, \dots, x_n, y_{n-b}, \dots, y_n)$, $b \gg 0$.

$P_b \in \mathbb{Z}[X_0, \dots, X_b, Y_0, \dots, Y_b]$ gives the addition of Witt vectors

$$\sum_{n\geq 0} V^n[X_n] + \sum_{n\geq 0} V^n[Y_n] = \sum_{n\geq 0} V^n[P_n(X_0, \dots, X_n, Y_0, \dots, Y_n)]$$

Problem of the unipotent Witt covectors: $\mathrm{Hom}(\underbrace{\mu_{p^\infty}}_{\text{not unipotent}}, CW^u) = 0$

$\rightarrow$ need to enlarge CW^u to classify all BT's that are not unipotent $\rightarrow t \in (B^+_{cris})^{\varphi=p}$ Fontaine's $2i\pi$ period of μ_{p^∞}

<u>Fontaine:</u> CW Witt covectors

$R = \mathbb{F}_p$-algebra

$$CW(R) = \left\{ [x_n]_{n \leq 0} \;\middle|\; \begin{array}{l} x_n \in R \text{ and } \exists N \leq 0 \text{ s.t.} \\ (x_n)_{n \leq N} \text{ is a nilpotent} \\ \text{ideal in } R \end{array} \right\}$$

Well defined group (Fontaine):

if $[x_n]_{n \leq 0}, [y_n]_{n \geq 0} \in CW(R)$ then

$\forall n$, the sequence $\left(P_b(x_{n-b}, \dots, x_n, y_{n-b}, \dots, y_n)\right)_{b \geq 0}$

is constant for $b >> 0$.

$CW^u \subset CW_{/\mathbb{F}_p}$

$\circlearrowleft$

F, V

$$V.[\ldots, x_{-1}, x_0] = [\ldots, x_{-2}, x_{-1}]$$

$$F.[x_m]_{m \leq 0} = [x_m^p]_{m \leq 0}.$$

Then: $\boxed{\mathbb{D}(H) = \mathrm{Hom}_k(H, CW_k)}$ for $H \in BT_k$

(Some kind of "Pontryagin duality" ~~[illegible]~~

$A \mapsto \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$)

$\mathbb{D}(H) \circlearrowleft F, V$ via the action on CW

Then if $M = \mathbb{D}(H)$ one finds back H via

$$\boxed{H = \mathrm{Hom}_{F,V}(M, CW)}$$

Examples:

* $M = W(k) \cdot e$ $\quad$ $Fe = e$, $\quad V e = pe$ $\qquad R = \mathbb{F}_p$-algebra

$\mathrm{Hom}_{F,V}(M, CW(R))$

$= \{ [x_n]_{n \leq 0} \,/\, x_n \in R,\ \sum_{n \leq N} R x_n$ nilpotent for $N \ll 0$

and $x_n^p = x_n \}$

$x_n = 0$ for $n \ll 0$

$= \underline{\mathbb{Q}_p/\mathbb{Z}_p}(R)$

$\Rightarrow$ $M = \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$, $\quad \mathbb{Q}_p/\mathbb{Z}_p = \{ [x_n]_{n \leq 0} \in CW \,/\, x_n^p = x_n \}$

* $M = W(k) \cdot e$ $\quad$ $F.e = pe$, $\quad V.e = e$.

$$\mathrm{Hom}_{F,V}(M, CW(R)) = \left\{ [x_n]_{n \leq 0} \;\middle|\; \begin{matrix} x_n \in R \\ x_{n-1} = x_n \\ x_n \text{ nilpotent} \end{matrix} \right\}$$

$$= \widehat{\mathbb{G}}_m(R)$$

$$\widehat{\mathbb{G}}_m \xrightarrow{\simeq} CW^{V=Id}$$

$$x \longmapsto \sum_{n \geq 0} V^n[x]$$

* d, h $\lambda = \frac{d}{h} \in]0,1]$ $\quad d \geq 1$, $(d,h) = 1$

$$H_\lambda = \ker\left(CW \xrightarrow{V^d - F^{h-d}} CW\right)$$

formal p-divisible groupe slope λ.

$$= \left[\dots, z_{d-1}^{p^{h-d}}, \dots, z_0^{p^{h-d}}, z_{d-1}, \dots, z_0\right] \in CW$$

$z_0, \dots, z_{d-1}$ nilpotent

$$H_\lambda = \mathrm{Spf}\left(k[[z_0, \dots, z_{d-1}]]\right).$$